The Dirichlet Laplacian Eigenfunction Problem

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This is a self-contained notes on the eigenvalue problem of the Laplace operator with Dirichlet boundary conditions. Basic functional analysis is assumed, but no formal knowledge of partial differential equations is required. We focus mostly on the computational aspects of the eigenfunction problem and work only in the Euclidean space.

The Laplacian eigenfunction problem aims to find the **eigenfunctions** $\{\phi_j\}$ of the **Laplacian** Δ on a domain Ω while satisfying the **Dirichlet boundary conditions** of eigenfunctions vanishing on the boundary $\partial\Omega$ of the domain. Laplacians with Dirichlet boundary conditions are often simply called **Dirichlet Laplacians**. The formal set of equations, with the first line also called the **Helmholtz equation**, is given below.

$$\begin{cases} -\Delta\phi_j(x) = \lambda_j\phi_j(x), & x \in \Omega, \\ \phi_j(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

For notational simplicity, we would often write the operator $-\Delta$ as L, and the sign of the Laplacian can go either way depending on the context.

1 Motivations

Eigenfunctions of the Laplacian with Dirichlet boundary conditions are used in the study of the boundary value problem – a special type of partial differential equations – by reformulating the original problem in the function space to the coefficient space of the eigenfunctions.

The eigenfunction problem is also studied in differential geometry, where one considers the Laplace-Beltrami operator of a Riemannian manifold and uses the information about its spectrum (the collection of eigenvalues) to inform the geometric structure.

Recently, the eigenfunction problem has been considered in machine learning for Gaussian process regression, where the covariance function of the Gaussian process is linked to the Laplacian, and finitely many eigenfunctions are used to approximate it while potentially admitting boundary conditions.

Here, we merely consider finding the eigenfunctions of a Dirichlet Laplacian, and (mostly) omit the discussions on subsequent usage of the found functions.

2 Simple Boundaries

In this section, we will solve the Laplacian eigenfunction problem when our boundaries are simple enough for exact, analytic solutions to exist. In particular, we will look at one one-dimensional interval, two two-dimensional rectangles, and discs as the three considered scenarios.

2.1 Spectral Properties of the Dirichlet Laplacian

We start by establishing some properties about the eigenfunction and eigenvalues for the Laplacian $L = -\Delta$ with Dirichlet boundary conditions. The collection of eigenvalues is also called the **spectrum**, and thus the properties below are denoted as the spectral properties. We only show the results in $\mathbb{R}$ and $\Omega = (0, 1)$[1] here.

We will show that, for $-\Delta$ with Dirichlet boundary conditions, the eigenvalues λ_j are *positive real* numbers, and the eigenfunctions ϕ_j are *real-valued*. Furthermore, the eigenfunctions are *orthogonal*.

[1]The exact upper and lower bounds of Ω are irrelevant, so we set it to the unit interval for simplicity, WLOG.

Consider our operator of interest $L = -\Delta : C^2[0,1] \to C[0,1]$ that maps a twice-continuously differentiable function to a continuous function. When we also consider the boundary condition, the domain of the operator is restricted to those that take zero at $\{0,1\}$, and we refer to the boundary-aware operator as L_D.

First, L_D is an symmetric operator for L_2 inner product $\langle \cdot, \cdot \rangle$, i.e. $\langle L_D f, g\rangle = \langle f, L_D g\rangle$ for any f, g in the domain of L_D. To show this, we have

$$\begin{aligned}
\langle L_D f, g\rangle &= \int_0^1 -f''(x)\overline{g(x)}dx \\
&= \left[-f'(x)\overline{g(x)}\right]_0^1 + \int_0^1 \overline{g}'(x)f'(x)dx \\
&= \left[-f'(x)\overline{g(x)}\right]_0^1 + [f(x)\overline{g}'(x)]_0^1 - \int_0^1 \overline{g}''(x)f(x)dx \\
&= -\int_0^1 f(x)\overline{g}''(x)dx = \langle f, L_D g\rangle
\end{aligned}$$

for complex conjugate $\bar{\cdot}$, where we performed integration by parts twice and noticed that f, g are in the domain of L_D so vanish at $\{0,1\}$.

Next, we can show that we do not have to worry about complex numbers, as the eigenvalues and eigenfunctions are real. For any non-zero function ϕ and corresponding constant λ, we have

$$\begin{aligned}
\langle L_D\phi, \phi\rangle &= \langle \lambda\phi, \phi\rangle = \lambda\langle\phi,\phi\rangle \\
\langle \phi, L_D\phi\rangle &= \langle \phi, \lambda\phi\rangle = \overline{\lambda}\langle\phi,\phi\rangle \\
\langle L_D\phi, \phi\rangle &= \langle \phi, L_D\phi\rangle \\
\lambda &= \overline{\lambda}
\end{aligned}$$

so $\lambda \in \mathbb{R}$. Next, for λ, its corresponding non-zero eigenfunction ϕ can be decomposed into $\phi = \phi_1 + i\phi_2$. Then, we have

$$\begin{aligned}
L_D\phi &= \lambda\phi \\
L_D\phi &= L_D(\phi_1 + i\phi_2) \\
&= L_D\phi_1 + iL_D\phi_2 \\
\lambda\phi &= \lambda(\phi_1 + i\phi_2) \\
&= \lambda\phi_1 + i\lambda\phi_2
\end{aligned}$$

so

$$\begin{cases} L_D\phi_1 & = \lambda\phi_1 \\ L_D\phi_2 & = \lambda\phi_2. \end{cases}$$

While ϕ is non-zero, ϕ_1 and ϕ_2 cannot both be zero, so either can be used in isolation as the real-valued eigenfunction corresponding to λ. Therefore, there is always a real-valued eigenfunction ϕ corresponding to eigenvalue $\lambda \in \mathbb{R}$.

Then, we consider two eigenvalues λ_1, λ_2 and their corresponding eigenfunctions ϕ_1, ϕ_2. We have

$$\begin{aligned}
\lambda_1\langle\phi_1,\phi_2\rangle &= \langle\lambda_1\phi_1,\phi_2\rangle = \langle L_D\phi_1,\phi_2\rangle \\
&= \langle\phi_1, L_D\phi_2\rangle = \langle\phi_1,\lambda_2\phi_2\rangle \\
&= \lambda_2\langle\phi_1,\phi_2\rangle.
\end{aligned}$$

This means, as ϕ_1, ϕ_2 are distinct, the only way for the above equality to hold is for $\langle\phi_1,\phi_2\rangle = 0$, which means any pair of eigenfunctions is orthogonal.

Finally, for eigenvector λ and its non-zero eigenfunction ϕ that we have normalised so $\langle\phi,\phi\rangle = 1$, we have

$$\begin{aligned}
\lambda &= \lambda\langle\phi,\phi\rangle = \langle\lambda\phi,\phi\rangle \\
&= \langle L_D\phi,\phi\rangle = \int_0^1 -\phi''(x)\phi(x)dx \\
&= [\phi(x)\phi(x)]_0^1 + \int_0^1 \phi'(x)\phi'(x)dx \\
&= \int_0^1 [\phi'(x)]^2 dx \geq 0
\end{aligned}$$

where we used integration by parts. Note that the above equality only holds when ϕ' is identically zero, which means ϕ is a constant. Since ϕ is in the domain of L_D and satisfies the boundary condition, the equality above only holds for $\phi = 0$, which contradicts the fact that ϕ is non-zero. Therefore, all eigenvalues λ are positive.

2.2 One-Dimensional Interval

Consider the eigenvalue problem (1) on an one-dimensional interval $\Omega = (0, l)$ for positive constant $l > 0$ so the boundary $\partial\Omega = \{0, l\}$.

Recall from Section 2.1 that the eigenvalues are positive real numbers, we reparametrise it by $\lambda =: \theta^2$ so the equations become

$$\phi''(x) + \theta^2\phi(x) = 0.$$

Now, this is a second-order ODE, which can be solved by the general solution $\phi(x) = c_1 \cos(\theta x) + c_2 \sin(\theta x)$ by noticing the characteristic roots are $\pm\theta i$. Then, using the boundary conditions, we have $\phi(0) = c_1 = 0$ and

$$\phi(l) = c_2 \sin(\theta l) = 0.$$

Since ϕ is non-zero, $c_2 \neq 0$ and we must have $\sin(\theta l) = 0$ which means $\theta = \pm j\pi$ for $j = 1, 2, \ldots$. Thus, the eigenvalues and the corresponding (normalised) eigenfunctions are, for $j = 1, 2, \ldots$,

$$\boxed{\lambda_j = \frac{j^2\pi^2}{l^2}, \quad \phi_j(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{j\pi}{l}x\right).}$$

2.3 Two-Dimensional Rectangle

Consider the eigenvalue problem (1) on a two-dimensional rectangle $\Omega = (0, l_1) \times (0, l_2)$ for positive constants $l_1, l_2 > 0$ so the boundary $\partial\Omega$ is the four edges of the rectangle.

This problem is very similar to that of Section 2.2. We start by guessing the solution is separable in dimensions, i.e. $\phi(x) = \phi_1(x_1)\phi_2(x_2)$, and solve that. This, as it turns out, would yield a collection of eigenfunctions that are sufficient in the sense that every continuous function supported on the closure of the domain can be approximated arbitrarily well using a weighted sum of these eigenfunctions.

Plugging our separable ansatz into (1) yields

$$\begin{aligned} -\phi_1''(x_1)\phi_2(x_2) - \phi_1(x_1)\phi_2''(x_2) &= \lambda\phi_1(x_1)\phi_2(x_2) \\ -\phi_1''(x_1)\phi_1^{-1}(x_1) - \phi_2''(x_2)\phi_2^{-1}(x_2) &= \lambda \end{aligned}$$

for any $x = (x_1, x_2) \in \Omega$. This means, both $\phi_1''(x_1)\phi_1^{-1}(x_1)$ and $\phi_2''(x_1)\phi_2^{-1}(x_1)$ are constant functions, as differentiating the above equation with either x_1 or x_2 will reveal. Then, if we set θ_1 and θ_2 such that $\theta_1 + \theta_2 = \lambda$ with

$$-\phi_1''(x_1)\phi_1^{-1}(x_1) = \theta_1, \qquad -\phi_2''(x_2)\phi_2^{-1}(x_2) = \theta_2,$$

then we recover two copies of the eigenfunction problem in Section 2.2. Therefore, we have

$$\theta_{1,m} = \frac{m^2\pi^2}{l_1^2}, \qquad \phi_{1,m}(x_1) = \sqrt{\frac{2}{l_1}} \sin\left(\frac{m\pi}{l_1}x_1\right)$$

and

$$\theta_{2,n} = \frac{n^2\pi^2}{l_2^2}, \qquad \phi_{2,n}(x_2) = \sqrt{\frac{2}{l_2}} \sin\left(\frac{n\pi}{l_2}x_2\right)$$

giving us the overall result

$$\boxed{\lambda_{m,n} = \frac{m^2\pi^2}{l_1^2} + \frac{n^2\pi^2}{l_2^2}, \qquad \phi(x) = \frac{2}{\sqrt{l_1 l_2}} \sin\left(\frac{m\pi}{l_1}x_1\right) \sin\left(\frac{n\pi}{l_2}x_2\right).}$$

2.4 Two-Dimensional Disc

Consider the eigenvalue problem (1) on a two-dimensional disc $\Omega = \{(x_1, x_2) \mid x_1^2 + x_2^2 < A^2\}$ for some constant $A > 0$. The boundary $\partial\Omega$ is then the circle characterised by $x_1^2 + x_2^2 = A^2$.

The separation of variable approach of Section 2.3 is not going to be immediately helpful here, as the boundary condition is not separable. However, we can reparameterise the problem using polar coordinates, then use separability.

First, we need to change the variables and compute how $-\Delta$ is going to become in the polar coordinates. We set $\phi(x_1, x_2) = \psi(r, \theta)$. Immediately, we have

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan(x_2/x_1), \quad \Omega = \{r < A, -\pi \le \theta < \pi\}$$

and

$$\begin{aligned}
\frac{\partial r}{\partial x_1} &= x_1(x_1^2 + x_2^2)^{-1/2} = r\cos\theta/r = \cos\theta \\
\frac{\partial r}{\partial x_2} &= x_2(x_1^2 + x_2^2)^{-1/2} = r\sin\theta/r = \sin\theta \\
\frac{\partial \theta}{\partial x_1} &= \cos^2(\theta)(-x_2 x_1^{-2}) = -\cos^2(\theta)\sin(\theta)r^{-1}\cos^{-2}(\theta) = -\frac{\sin\theta}{r} \\
\frac{\partial \theta}{\partial x_2} &= \cos^2(\theta)x_1^{-1} = \cos^2(\theta)r^{-1}\cos^{-1}(\theta) = \frac{\cos\theta}{r}.
\end{aligned}$$

Using chain rules would give us

$$\begin{aligned}
\frac{\partial \phi}{\partial x_1} &= \frac{\partial \psi}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial x_1} = \cos\theta\frac{\partial \psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \psi}{\partial \theta} \\
\frac{\partial \phi}{\partial x_2} &= \frac{\partial \psi}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial x_2} = \sin\theta\frac{\partial \psi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \psi}{\partial \theta}
\end{aligned}$$

and thus the second derivatives are, omitting derivations,

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial x_1^2} &= \cos^2(\theta)\frac{\partial^2 \psi}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 \psi}{\partial\theta\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial \psi}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial \psi}{\partial \theta} \\
\frac{\partial^2 \phi}{\partial x_2^2} &= \sin^2(\theta)\frac{\partial^2 \psi}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 \psi}{\partial\theta\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial \psi}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial \psi}{\partial \theta}
\end{aligned}$$

which can be combined to give the Laplacian in the polar coordinate, denoted by Δ_p,

$$-\Delta_p\psi = -\frac{\partial^2 \psi}{\partial r^2} - r^{-2}\frac{\partial^2 \psi}{\partial \theta^2} - r^{-1}\frac{\partial \psi}{\partial r}.$$

Next, we are ready to impose separation of variables to solve the eigenfunction problem for discs. We set $\psi(r, \theta) = \psi_1(r)\psi_2(\theta)$. Also, the boundary conditions on x_1, x_2 are now imposed on r, θ, with $\psi_1(A) = 0$ and $\psi_2(\pi) = \psi_2(-\pi)$. So, we have

$$\begin{aligned}
\lambda\psi_1(r)\psi_2(\theta) &= -\phi_1''(r)\psi_2(\theta) - r^{-2}\psi_1(r)\psi_2''(\theta) - r^{-1}\psi_1'(r)\psi_2(\theta) \\
\psi_2^{-1}(\theta)\psi_2''(\theta) &= -r^2\psi_1''(r)\psi_1^{-1}(r) - r\psi_1'(r)\psi_1^{-1}(r) - \lambda r^2
\end{aligned}$$

which means

$$\begin{aligned}
\psi_2^{-1}(\theta)\psi_2''(\theta) &= \gamma \\
-r^2\psi_1''(r)\psi_1^{-1}(r) - r\psi_1'(r)\psi_1^{-1}(r) - \lambda r^2 &= \gamma
\end{aligned}$$

for some γ. We can solve ψ_2 easily and note that the eigenvalues would be of the form $\gamma = n^2$ for $n = 1, 2, \ldots,$ and correspond to two independent eigenfunctions $\cos(n\theta)$ and $\sin(n\theta)$. For eigenvalue $n = 0$, the eigenfunction of ψ_2 is ca onstant function 1.

Then, for ψ_1, we have

$$r^2\psi_1''(r) + r^1\psi_1'(r) + (\lambda r^2 - \gamma)\psi_1(r) = 0$$

after some rearrangement, which is the Bessel equation. The solution is the Bessel function $J_n(\sqrt{\lambda}r)$ of order n, defined as

$$J_n(s) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{s}{2}\right)^{2j+n}.$$

with roots $s_{n,m}$ of $m = 1, 2, \dots$ that do not admit simple formula.

Therefore, we have obtained the doubly indexed sequence of eigenvalues for $-\Delta_p$:

$$\boxed{\begin{aligned} &\lambda_{m,0} = \frac{s_{0,m}^2}{A^2}, \quad \psi_{m,0}(r,\theta) = J_0(s_{0,m}r/A) \\ &\lambda_{m,n} = \frac{s_{n,m}^2}{A^2}, \quad \psi_{m,n}^{(1)}(r,\theta) = J_n(s_{n,m}r/A)\cos(n\theta), \quad \psi_{m,n}^{(2)}(r,\theta) = J_n(s_{n,m}r/A)\sin(n\theta) \end{aligned}}$$

for $m, n = 1, 2, \dots$.

3 General Boundaries – Finite Difference

Solving the Laplacian eigenfunction problem when the boundaries are no longer as simple as in the cases of Section 2 means we have to resort to approximate, numerical solutions. In this notes, we look at two common numerical approaches – the finite difference in this section and the finite element in the next section – of solving the eigenfunction problem. Both approaches approximate the infinite-dimensional problem by a finite-dimensional one, so the eigenfunction problem becomes an eigenvector problem, which can then be solved using linear algebra.

The derivative of a univariate function f at a point x is commonly approximated using the definition

$$\frac{d}{dx}f(x) \approx \frac{1}{h}[f(x+h) - f(x)]$$

for some small discretisation parameter $h > 0$. The second derivative is then approximated using the above approximation twice, so

$$\frac{d^2}{dx^2}f(x) \approx \frac{f'(x) - f'(x-h)}{h} \approx \frac{[f(x+h) - f(x)] - [f(x) - f(x-h)]}{h^2} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

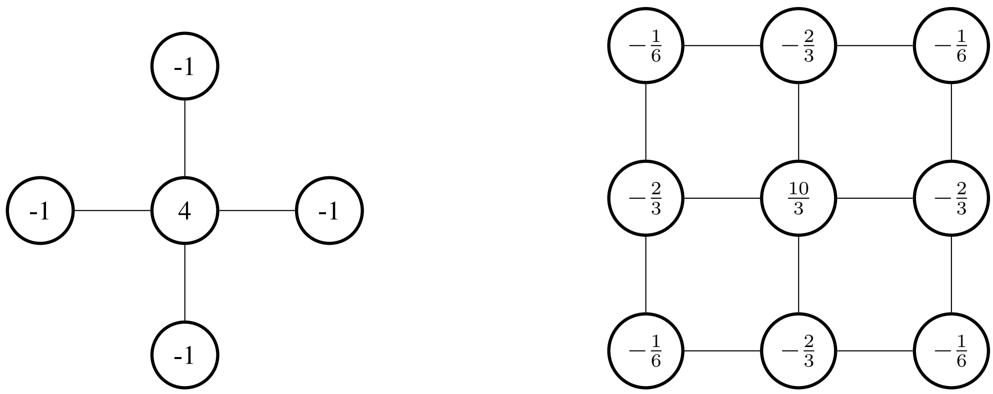
which is typically called the **second-order central difference** as the reference point is set to be x itself. These approximations give us the **finite difference** method to numerically solve the Dirichlet Laplace equation and many more.

The differential operator in question in (1) is the Laplacian, and naturally, we can use the above central difference approximation to discretise the Laplacian. We will be working in $\mathbb{R}^2$ for the ease of exposition, but it is possible and straightforward to extrapolate to other dimensions. So, we have, using $\boldsymbol{x} := (x_1, x_2)$,

$$\begin{aligned} \Delta f(x_1, x_2) &= \partial_{x_1x_1} f(\boldsymbol{x}) + \partial_{x_2x_2} f(\boldsymbol{x}) \\ &\approx \frac{1}{h^2}\left[f(x_1+h, x_2) - 2f(x_1, x_2) + f(x_1-h, x_2) + f(x_1, x_2+h) - 2f(x_1, x_2) + f(x_1, x_2-h)\right] \\ &= \frac{1}{h^2}\left[f(x_1+h, x_2) + f(x_1-h, x_2) + f(x_1, x_2+h) + f(x_1, x_2-h) - 4f(x_1, x_2)\right] \end{aligned}$$

which approximates each point's Laplacian by the sum of its immediate neighbours, subtracting four times itself, and weighting by h^{-2}. This is often called the **5-point stencil**, and is illustrated below in Figure 1a. We can also conduct a higher-order discretisation to improve the accuracy and resolve some anisotropy issues of the 5-point method, which motivates the commonly used 9-point stencil of Figure 1b.

Motivated by the above Laplacian discretisation, we impose a regular square grid of discretisation parameter h with origin (x_1, x_2) over Ω, so the grid is $G = \{(x_1 + nh, x_2 + mh) \mid n, m \in \mathbb{Z}\}$. For now, we assume the region Ω is square with (x_1, x_2) as its centre and width as an even integer multiple of h, so there are grid points on the sides and corners. The region is therefore discretised using the grid to yield $\Omega_h := G \cap \Omega$ that intersects the grid and the region, and $\overline{\Omega}_h := G \cap \overline{\Omega}$ including points on the boundary. We will return to non-regular regions Ω later in Section 3.1.



(a) 5-Point Stencil

(b) 9-Point Stencil

Figure 1: Two copies of the 5-point stencil

Our Dirichlet Laplacian eigenfunction problem (1) is thus discretised both via the Laplace operator and the space. The discretised space Ω_h consists of points which we index by n, m so $\boldsymbol{x}_{n,m} = (x_1 + nh, x_2 + mh)$ with our grid origin (x_1, x_2). Then, we have

$$\begin{cases} h^{-2}[4\phi(\boldsymbol{x}_{n,m}) - \phi(\boldsymbol{x}_{n-1,m}) - \phi(\boldsymbol{x}_{n+1,m}) - \phi(\boldsymbol{x}_{n,m-1}) - \phi(\boldsymbol{x}_{n,m+1})] = \lambda\phi(\boldsymbol{x}_{n,m}), & \boldsymbol{x}_{n,m} \in \Omega, \\ \phi(\boldsymbol{x}_{n,m}) = 0, & \boldsymbol{x}_{n,m} \in \partial\Omega \end{cases}$$

which can be aggregated into the following linear equation

$$Au = \lambda u$$

where u is a column vector of $\phi(\boldsymbol{x}_{n,m})$ re-indexed by i (arbitrarily, e.g. lexicographic order, red-black ordering) and A is the constant matrix constructed as follows:

- If $\boldsymbol{x}_i$ is on the boundary $\partial\Omega$, then $A_{i,i} = 1$.
- If $\boldsymbol{x}_i$ and all four immediate neighbourhoods are in Ω, then $A_{i,i} = 4$ and its neighbour indexed j has $A_{i,j} = -1$.
- If $\boldsymbol{x}_i$ is in Ω but a neighbour indexed j is on the boundary $\partial\Omega$, then $A_{i,j} = 0$.

Thus, we obtain the finite difference approximation to the original problem (1) and turn the problem into an eigenvector problem from linear algebra

$$Au = \lambda u$$

which can be solved using any standard eigen-decomposition tools.

3.1 Non-Regular Boundary

When the region Ω is not sufficiently regular, no discretisation parameter h can form a grid such that all points are either on the boundary or in Ω, say when Ω is a disc.

In such settings, we can incorporate the boundary by including new points and giving them values by interpolation. WLOG, we consider two horizontally consecutive points $(x_1 - h, x_2), (x_1, x_2)$ of a grid such that one of them is in Ω and the other is not, e.g. $(x_1 - h, x_2) \notin \Omega, (x_1, x_2) \in \Omega$. Then, we can find some $s \in (0, 1]$ such that $(x_1 - sh, x_2) \in \partial\Omega$. We can include that point in the grid set, remove $(x_1 - h, x_2)$, and set its function value $\phi(x_1 - sh, x_2) = 0$.

Afterwards, the finite difference at the point $(x_1 + h, x_2)$ using 5-point stencil while all but the left neighbour (x_1, x_2) in Ω becomes

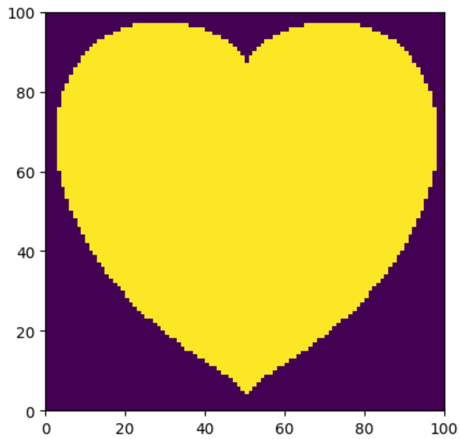
$$\begin{aligned} s^{-1}(s+1)^{-1}h^{-2}\left[(s+1)\phi(x_1, x_2) - s\phi(x_1 + h, x_2) - \phi(x_1 - sh, x_2)\right] \\ + h^{-2}\left[2\phi(x_1, x_2) - \phi(x_1, x_2 - h) - \phi(x_1, x_2 + h)\right] = 0 \end{aligned}$$

by adjusting the stepsize along the x direction accordingly. The rest follows as before, and we still yield a linear system of $Au = \lambda u$ but with slightly adjusted A and choice of points. One could imagine similar constructions for other types of boundary points, and how to do finite difference for the neighbouring points.

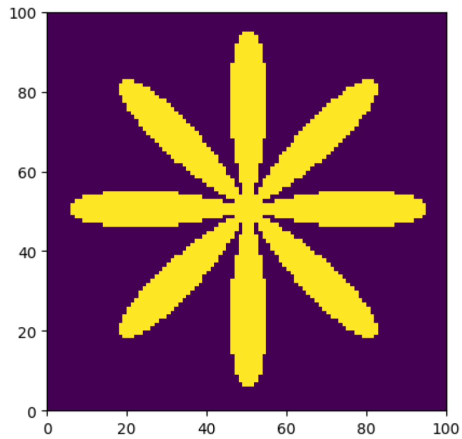
It should be apparent that, although feasible, resolving non-regular boundaries requires a lot of extra work and care. In practice, it is often simpler to ignore the boundary irregularity and set a smaller discretisation parameter h with a regular grid. This challenge with finite difference is also a key motivation for the finite element method that resolves irregular boundaries more naturally, which we will introduce in Section 4.

3.2 Examples: Heart and Star

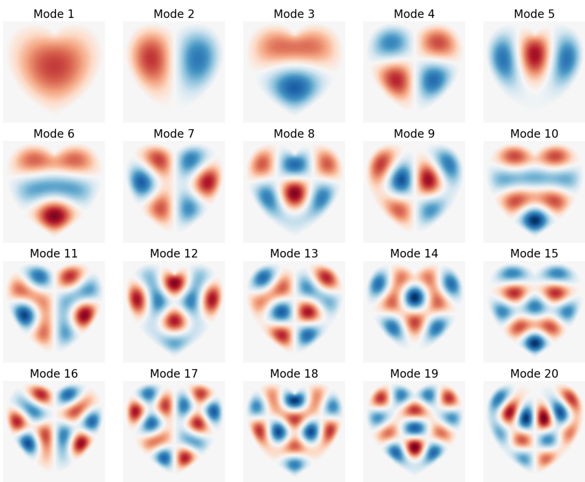
Two examples of the Dirichlet Laplacian eigenfunction problem (1) solved using the finite difference method on Ω being a heart and a star in 2D are conducted below. We use both 5-point (Figure 1a) and 9-point (Figure 1a) methods, and no special treatment is taken for the non-regular boundaries. A discretisation parameter of $h = 1$ is selected, and the region is bounded within $[0, 100] \times [0, 100]$ square – so equivalent to having $h = 0.01$ with a unit square.



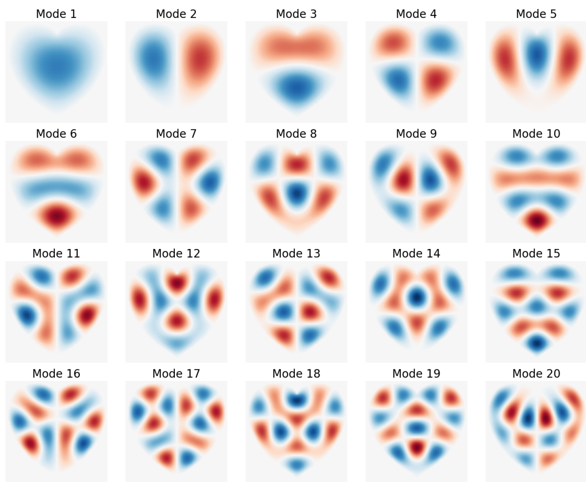
(a) Heart Boundary



(b) Star Boundary



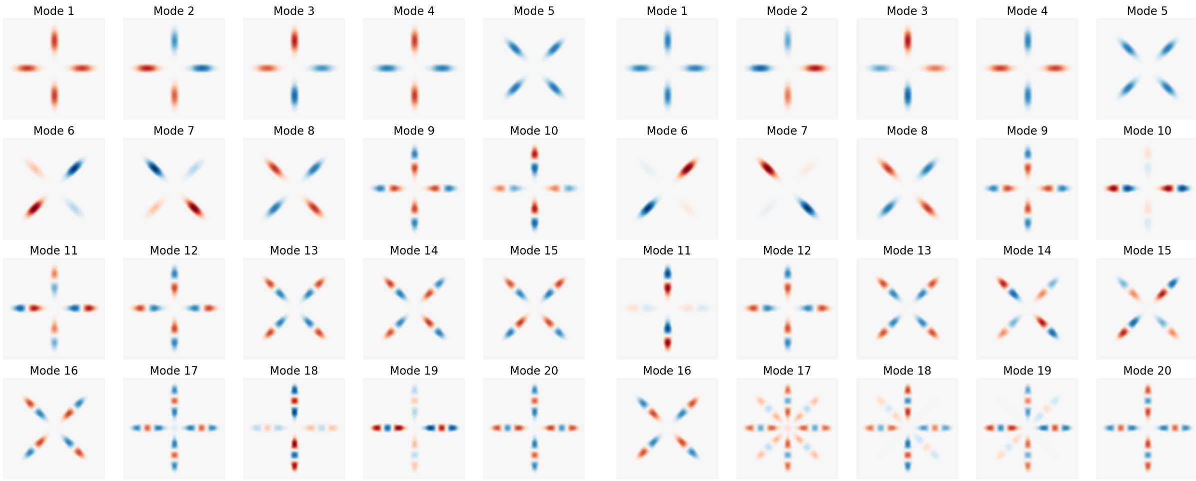
(a) Heart Eigenfunctions with 5-Point Stencil



(b) Heart Eigenfunctions with 9-Point Stencil

4 General Boundaries – Finite Element

Another commonly used algorithm to numerically solve differential equations is the **finite element method**, and it is especially good at resolving the geometry of the considered region Ω. The original Laplacian eigenfunction problem (1) seeks twice-differentiable solutions $\phi \in C^2(\Omega)$, while the finite element method (and the more general **Galerkin method**) searches for approximate solutions that best satisfy (1) in a finite-dimensional function space $V_m(\Omega)$ with basis $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$. The finite-dimensional approximation (or projection) can conveniently convert an infinite-dimensional problem to a finite-dimensional one to apply existing tools from linear algebra. These points will be made precise in the following.



(a) Star Eigenfunctions with 5-Point Stencil (b) Star Eigenfunctions with 9-Point Stencil

To begin, we consider the *weak / variational* formulation of (1), which is given by integrating over any *test function* $v \in V$. V is to be specified later, think for now of a sufficiently large and differentiable function class that allows integration by parts and vanishes at $\partial\Omega$. So we have

$$\text{Find } \phi \text{ s.t.} \qquad \int_\Omega -\Delta\phi(x)v(x)dx = \int_\Omega \lambda\phi(x)v(x)dx.$$

Using Green's first identity (multivariate version of integration by parts), we have

$$\int_\Omega -\Delta\phi(x)v(x)dx = [-\nabla\phi \cdot v]_\Omega + \int_\Omega \nabla\phi(x)\cdot\nabla v(x)dx = \int_\Omega \nabla\phi(x)\cdot\nabla v(x)dx$$

where the first term vanishes by the construction of V. This means, our original problem (1) is phrased in the weak form of

$$\int_\Omega \nabla\phi(x)\cdot\nabla v(x)dx = \lambda\int_\Omega \phi(x)v(x)dx \qquad \forall v \in V.$$

Given V, we then construct a finite-dimensional, closed subspace V_m with basis $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ and find the approximate solution $\phi^m \in V_m$ – also called the *trial function* – that satisfies

$$\int_\Omega \nabla\phi^m(x)\cdot\nabla v(x)dx = \lambda\int_\Omega \phi^m(x)v(x)dx \qquad \forall v \in V_m.$$

Since $\phi^m \in V_m$, we can decompose it as

$$\phi^m(x) = \sum_{i=1}^m u_i\varphi_i(x)$$

for coefficients u_i. Plugging this expression into the weak form yields

$$\begin{aligned}
\int_\Omega \nabla\phi^m(x)\nabla v(x)dx &= \int_\Omega \nabla\left[\sum_{i=1}^m u_i\varphi_i(x)\right]\nabla v(x)dx \\
&= \int_\Omega \left[\sum_{i=1}^m u_i\nabla\varphi_i(x)\right]\nabla v(x)dx \\
&= \sum_{i=1}^m u_i \int_\Omega \nabla\varphi_i(x)\nabla v(x)dx; \\
\lambda\int_\Omega \phi^m(x)v(x)dx &= \lambda\int_\Omega \left[\sum_{i=1}^m u_i\varphi_i(x)\right]v(x)dx \\
&= \lambda\sum_{i=1}^m u_i\int_\Omega \varphi_i(x)v(x)dx; \\
\sum_{i=1}^m u_i\int_\Omega \nabla\varphi_i(x)\nabla v(x)dx &= \lambda\sum_{i=1}^m u_i\int_\Omega \varphi_i(x)v(x)dx.
\end{aligned}$$

As the weak form works for any $v \in V_m$, we can set v as $\varphi_1, \varphi_2, \ldots, \varphi_m$ so that once combined we have a linear system of equations

$$Au = \lambda Mu$$

with matrices A, M where entries are computable via

$$A_{j,i} := \int_\Omega \nabla\varphi_i(x)\nabla\varphi_j(x)dx, \qquad M_{j,i} := \int_\Omega \varphi_i(x)\varphi_j(x)dx.$$

Therefore, we can get the finite element approximate solution u by solving the generalised eigenvector problem $Au = \lambda Mu$. The matrices A, M are sometimes called the **stiffness** and **mass** matrices, respectively.

To recap the description above, we carried out the following steps to establish the finite element method for the Dirichlet Laplacian eigenfunction problem:

1. Reformulate the problem into the weak form using test functions from the function space V.
2. Find a finite-dimensional subspace V_m of V with known basis functions.
3. Restrict the weak form by limiting the test function space to V_m to solve using linear algebra.

In the rest of this section, we will explore the choices of the basis $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$, which will be explored in Section 4.1, and present some concrete examples in Section 4.2.

4.1 Mesh Construction

The key discretisation of FEM is via replacing the infinite-dimensional function space V of test and solution functions by a finite-dimensional V_m with known basis functions. This differs from finite difference on a fundamental level – finite difference discretises (differential) operators, whereas finite element discretises functions directly.

The choice of V_m and its basis function is extremely flexible. For example, if one picks sinusoids with varying frequencies, the method is called the *spectral method*. Here, we will be considering piece-wise polynomial basis functions for the finite element method.

Effectively, one discretises the full considered region Ω into smaller pieces called *elements* using *nodes*. For example, in Figure 5, the considered region is an L shape, and we put nodes (marked dots) inside, which form triangular elements. The basis functions are then defined on those elements in a piecewise fashion. The created discretised region is also called the *mesh*.

For simplicity, we consider a one-dimensional interval $\Omega = [0, L]$ as our considered region and consider the nodes to be evenly distributed points $0 = x_0 < x_1 < \cdots < x_m = L$ with distance h (assume $m = L/h$ is an integer). This gives us subregions $\Omega_1 = [x_0, x_1], \Omega_2 = [x_1, x_2], \cdots, \Omega_m = [x_{m-1}, x_m]$ as elements. Then, one choice of the basis function would be the hat functions, where it is linearly increasing from zero to one over a

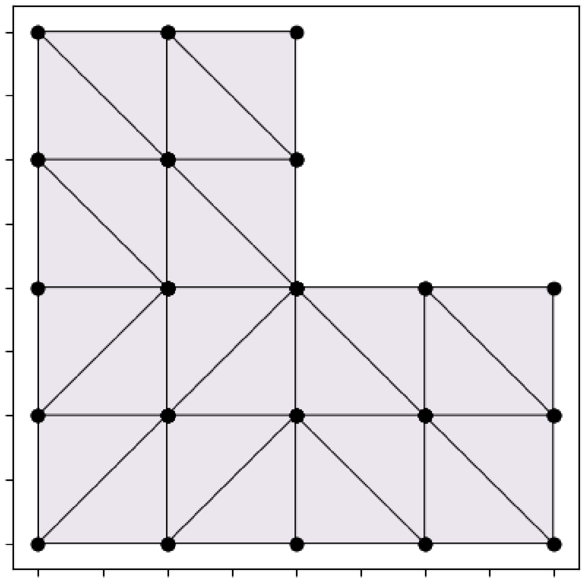


Figure 5: Mesh for an L-shaped region.

subregion and then linearly decreasing from one to zero over the next subregion while being zero everywhere else. Mathematically, this means a basis function φ_k that is non-zero only on Ω_k and Ω_{k+1}, and is defined as

$$\varphi_k(x) = \begin{cases} \frac{1}{h}(x - x_k) & x \in \Omega'_k \\ \frac{1}{h}(x_{k+1} - x) & x \in \Omega'_{k+1} \\ 0 & \text{elsewise.} \end{cases} \tag{2}$$

Note that here we assume the evenly distributed nodes with a gap h. If the distribution is uneven, then we replace h by indexed $h_k := x_k - x_{k-1}$. This gives a piece-wise polynomial basis with order $p = 1$. See Figure 6 for an illustration where $\Omega = [0, 4]$ and $h = 0.5$.

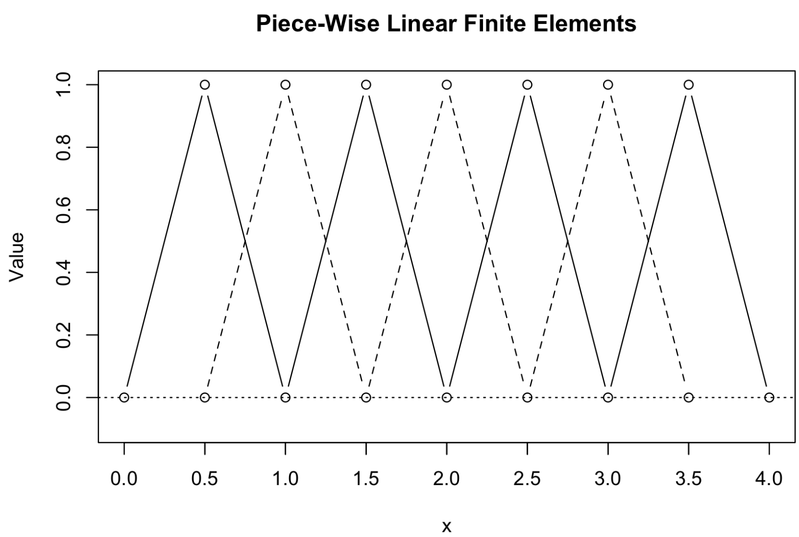


Figure 6: Hat function over an interval.

The hat function basis is particularly nice because it yields sparse stiffness and mass matrices A and M. Notice that for ϕ_i and ϕ_j with non-consecutive i, j (so $|i - j| > 1$), we have

$$\int_\Omega \phi_i(x)\phi_j(x)dx = 0, \qquad \int_\Omega \nabla\phi_i(x)\nabla\phi_j(x)dx = 0$$

due to disjoint support, thus most entries of A, M are zero. Furthermore, those non-zero entries are simple to calculate – details skipped as they are routine calculus. The sparsity can be heavily exploited when solving the linear system $Au = \lambda Mu$ for obtaining the FEM approximations to speed up the matrix computation.

To recap, we have suggested ways to break a region down into smaller pieces using nodes and their corresponding elements, and suggested a way to interpolate between them to form bases using piece-wise linear functions.

To improve the approximation, there are two immediate directions: (1) refine the mesh by reducing the gap h, (2) increase the interpolating function order p. There is an extensive literature on adaptive FEM where h or p or both are being improved sequentially to effectively boost solution quality. In general, adaptive FEM is a broad research area and a proper discussion of methods is beyond the scope of this notes.

4.2 Examples

We applied FEM to solve the Dirichlet Laplacian eigenfunction problem (1) with the heart boundary as in Section 3.2. The results are presented below with the produced eigenfunctions and the constructed mesh in Figure 7.

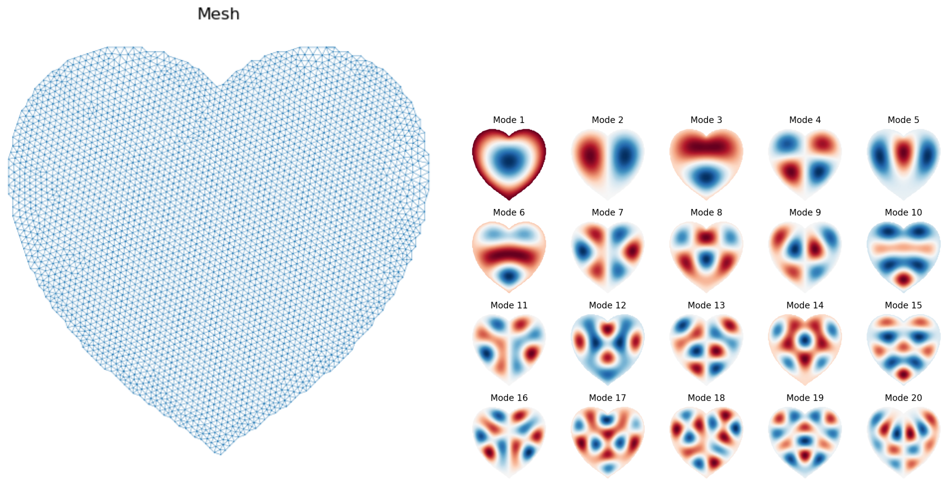


Figure 7: FEM solution to the heart boundary Dirichlet Laplacian eigenfunction problem.